Name concentration risk and its measurement

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Introduction

- It is very important for a bank to manage the risks originated from its business activities. The credit risk underlying the credit portfolio is often the largest risk in a bank.

- Basel Accords laid the basis for international minimum capital standards. Banks became subject to regulatory capital requirements.

- Basel II is structured in a three Pillar framework:
  - Pillar 1: more risk sensitive minimal capital requirements.
  - Pillar 2: banks are allowed to calculate the economic capital (risk concentration).
  - Pillar 3: transparency in bank’s financial reporting.
Introduction

Concentration risks arise from an unequal distribution of loans to single borrowers (exposure or name concentration) or different industry or regional sectors (sector concentration).

Within Basell II banks may opt for the standard approach (more conservative) or the internal rating based (IRB) approach (more advanced).

Merton model: basis of the Basel II IRB approach. Under homogeneity conditions, this model leads to the ASRF model. However, this model can underestimate risks in the presence of exposure concentration.

Credit risk managers are interested in:
- How can concentration risk be quantified?
- How can risk measures be accurately computed in short times?
- How does the individual transactions contribute to the total risk?
Risk Parameters

- We specify a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with filtration \((\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions. We fix a time horizon \(T > 0\) (usually one year).

- We consider a credit portfolio consisting of \(N\) obligors.

- Any obligor \(n\) is characterized by:
  - The **exposure at default** \(E_n\): potential exposure measured in currency.
  - The **loss given default** \(L_n\): magnitude of likely loss on the exposure as a percentage of the exposure.
  - The **probability of default** \(P_n\): likelihood that a loan will not be repaid.

Each of them can be estimated from empirical default data.
Consider an obligor $n$ subject to default in the fixed time horizon $T$. We introduce $D_n$, the default indicator of obligor $n$,

$$D_n = \begin{cases} 1, & \text{if obligor } n \text{ is in default,} \\ 0, & \text{if obligor } n \text{ is not in default,} \end{cases}$$

where $\mathbb{P}(D_n = 1) = P_n$ and $\mathbb{P}(D_n = 0) = 1 - P_n$.

Let $\mathcal{L}$ be the portfolio loss given by,

$$\mathcal{L} = \sum_{n=1}^{N} \mathcal{L}_n,$$

where $\mathcal{L}_n = E_n \cdot L_n \cdot D_n$. 
Credit risk can split in **Expected Losses** EL (which can be forecasted) and **Unexpected Losses** UL (more difficult to quantify).

**Assumption 1.1**

*The exposure at default $E_n$, the loss given default $L_n$ and the default indicator $D_n$ of an obligor $n$ are independent.*

Denote by $EL_n$ the expectation value of $L_n$, therefore,

$$EL = \mathbb{E}(L) = \sum_{n=1}^{N} E_n \cdot EL_n \cdot P_n.$$ 

Holding the $UL = \sqrt{\mathbb{V}(L)}$ as a risk capital for cases of financial distress might not be appropriate (peak losses can be very large when they occur).
Let $\alpha \in (0, 1)$ be a given confidence level, the $\alpha$-quantile of the loss distribution of $L$ in this context is called **Value at Risk** (VaR). Thus,

$$\text{VaR}_\alpha = \inf \{ l \in \mathbb{R} : \mathbb{P}(L \leq l) \geq \alpha \} = \inf \{ l \in \mathbb{R} : F_L(l) \geq \alpha \},$$

where $F_L$ is the cumulative distribution function of the loss variable $L$.

VaR is the measure chosen in the Basel II Accord ($\alpha = 0.999$) for the computation of capital requirement.

Another important risk measure is the so called economic capital $EC_\alpha$ for a given confidence level $\alpha$,

$$EC_\alpha = \text{VaR}_\alpha - EL.$$
Risk Measures: VaR

Drawbacks:

1. VaR gives no information about the severity of losses which occur with probability less than $1 - \alpha$.

2. VaR is not a coherent risk measure, since it is not sub-additive. If $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ then $\text{VaR}_\alpha(\mathcal{L}) \not\subseteq \text{VaR}_\alpha(\mathcal{L}_1) + \text{VaR}_\alpha(\mathcal{L}_2)$ (this fact contradicts the intuition of diversification benefits associated with merging portfolios).
Example

Consider two independent loans with default indicators following a Bernoulli distribution $B(1, p)$ with $0.006 \leq p < 0.01$ and exposures equal to 1. Define two portfolios A and B, each of them consisting of one unit of the above introduced loans. Then if we denote the corresponding portfolio losses by $L_A$ and $L_B$,

$$\text{VaR}_{0.99}(L_A) = \text{VaR}_{0.99}(L_B) = 0.$$ 

Now if we consider a portfolio C defined as the union of portfolios A and B and denote by $L_C = L_A + L_B$. Then,

$$\mathbb{P}(L_C = 0) = (1 - p)^2 < 0.99,$$

and therefore,

$$\text{VaR}_{0.99}(L_C) > 0,$$

so that,

$$\text{VaR}_{0.99}(L_C) > \text{VaR}_{0.99}(L_A) + \text{VaR}_{0.99}(L_B).$$
Risk Measures: Expected Shortfall

On the contrary the **Expected Shortfall** (ES) enjoys the coherence properties.

\[
ES_\alpha = \mathbb{E}(\mathcal{L} | \mathcal{L} \geq \text{VaR}_\alpha),
\]

or alternatively,

\[
ES_\alpha = \frac{1}{1 - \alpha} \int_{\text{VaR}_\alpha}^{+\infty} xf_\mathcal{L}(x)dx,
\]

\((f_\mathcal{L} \text{ density function of the loss variable } \mathcal{L}).\)
To decompose the risk measured by the VaR or the ES into individual transactions (allocation principle, D. Tasche (2000)) define the Risk Contribution to VaR (VaRC) of obligor $n$ at confidence level $\alpha$ by,

$$ \text{VaRC}_{\alpha,n} \equiv E_n \cdot \frac{\partial \text{VaR}_\alpha}{\partial E_n}, $$

and the Risk Contribution to ES (ESC) of obligor $n$ at confidence level $\alpha$ by,

$$ \text{ESC}_{\alpha,n} \equiv E_n \cdot \frac{\partial \text{ES}_\alpha}{\partial E_n}. $$

These definitions satisfy the additivity condition,

$$ \sum_{n=1}^{N} \text{VaRC}_{\alpha,n} = \text{VaR}_\alpha, \quad \sum_{n=1}^{N} \text{ESC}_{\alpha,n} = \text{ES}_\alpha. $$
Under appropriate conditions the marginal VaR contribution at confidence level $\alpha$ of the obligor $n$ is,

$$\text{VaRC}_{\alpha,n} = \mathbb{E}(L_n | L = \text{VaR}_\alpha),$$  

(1)

and the marginal contribution at confidence level $\alpha$ to the expected shortfall is,

$$\text{ESC}_{\alpha,n} = \mathbb{E}(L_n | L \geq \text{VaR}_\alpha).$$  

(2)
Credit risk models can be divided into two fundamental classes of models, **structural** or **asset-value models** and **reduced-form** or **default-rate models**.

Asset-value models have their origin on the famous **Merton** model, where the default of a firm is modeled in terms of the relationship between its assets and the liabilities that it faces at the end of a given time period.

Two famous industry models descending from the Merton approach are the **KMV** model (developed by Moody’s KMV) and the **CreditMetrics** model (developed by JPMorgan and the RiskMetrics Group).
General Framework

- Company’s debt is given by a zero-coupon bond with face value \( B \).
- The model also assumes that the asset value process \((V_t)_{t \geq 0}\) follows a geometric Brownian motion of the form,
  \[
  dV_t = \mu V_t dt + \sigma V_t dW_t, \quad 0 \leq t \leq T. \tag{3}
  \]
- The solution at time \( T \) of the stochastic differential equation (3) with initial value \( V_0 \) can be computed and is given by,
  \[
  V_T = V_0 e^{\left(\mu - \frac{1}{2} \sigma^2 V\right) T + \sigma V W_T}.
  \]
- This implies in particular that,
  \[
  \log V_T \sim N \left(\log V_0 + \left(\mu - \frac{1}{2} \sigma^2 V\right) T, \sigma^2 V T\right).
  \]
- The default probability of the firm by time \( T \) can be computed as,
  \[
  \mathbb{P}(V_T \leq B) = \mathbb{P}(\log V_T \leq \log B) = \Phi \left(\frac{\log \frac{B}{V_0} - \left(\mu - \frac{1}{2} \sigma^2 V\right) T}{\sigma \sqrt{T}}\right).
  \]
The Multi-Factor Merton Model

We define $V_{t}^{(n)}$ to be the asset value of the counterparty $n$ at time $t \leq T$. For every counterparty $\exists T_n$ s.t. counterparty $n$ defaults in the time period $[0, T]$ if $V_T^{(n)} < T_n$.

For $n = 1, \ldots, N$, we define,

$$D_n = \chi_{\{V_T^{(n)} < T_n\}} \sim B \left(1, \mathbb{P}(V_T^{(n)} < T_n)\right).$$

Consider the borrower $n$’s asset-value log return $r_n$: $\log \left(\frac{V_T^{(n)}}{V_0^{(n)}}\right)$.

Assumption 1.2

Asset returns $r_n$ depend linearly on $K$ standard normally distributed risk factors $X = (X_1, \ldots, X_K)$ affecting the borrowers’ defaults in a systematic way as well as on a standard normally distributed idiosyncratic term $\epsilon_n$. Moreover, $\epsilon_n$ are independent of the systematic factors $X_k$ for every $k \in \{1, \ldots, K\}$ and the $\epsilon_n$ are uncorrelated.
Under this assumption and after standardization,

\[ r_n = \beta_n Y_n + \sqrt{1 - \beta_n^2} \epsilon_n. \]

\( Y_n \) can be decomposed into \( K \) independent factors \( X = (X_1, \ldots, X_K) \) by,

\[ Y_n = \sum_{k=1}^{K} \alpha_{n,k} X_k, \quad \sum_{k=1}^{K} \alpha_{n,k}^2 = 1. \]

\( Y_n \) denotes the firm’s **composite factor** and \( \epsilon_n \) the **idiosyncratic shock**. \( \beta_n \) borrower \( n \)’s sensitivity to systematic risk \( Y_n \). \( \alpha_{n,k} \) describe the dependence of obligor \( n \) on a sector \( k \). 

Now we can rewrite equation (4) as,

\[ D_n = \chi\{r_n < t_n\} \sim B(1, \mathbb{P}(r_n < t_n)), \]

We have \( P_n = \mathbb{P}(r_n < t_n), \ t_n = \Phi^{-1}(P_n) \) and,

\[ P_n(y_n) \equiv \mathbb{P}(r_n < t_n | Y_n = y_n) = \Phi \left( \frac{t_n - \beta_n y_n}{\sqrt{1 - \beta_n^2}} \right), \text{ cond. default probability.} \]
Conditional Default Probability

Figure: Conditional default probabilities.
**Portfolio Loss**

**Purpose:** find an expression for the portfolio loss variable $\mathcal{L}$.

Assuming a constant loss given default equal to $L_n$ for obligor $n$, the portfolio loss distribution can then be derived as,

$$
\mathbb{P}(\mathcal{L} \leq l) = \sum_{(d_1,\ldots,d_N)\in\{0,1\}^N} \left( \sum_{n=1}^{N} s_n \cdot L_n \cdot d_n \right) \cdot \mathbb{P}(D_1 = d_1, \ldots, D_N = d_N).
$$

**Impractical** from a computational point of view for realistic portfolios (for instance $N = 1000$).

**Remark 1.1**

*We present an analytical approximation for the $\alpha^{th}$ percentile of the loss distribution in the one-factor framework, under the assumption that portfolios are infinitely fine-grained such that the idiosyncratic risk is completely diversified.*
Figure: Densities and distributions for portfolios A and B.
**The ASRF Model**

The Asymptotic Single Risk Factor Model (ASRF) is the model chosen in Basel II to calculate regulatory capital. It is based on the one-factor Merton model and it mainly relies in the following assumptions,

**Assumption 1.3**

1. **Portfolios are infinitely fine-grained, i.e. no exposure accounts for more than an arbitrarily small share of total portfolio exposure.**
2. **Dependence across exposures is driven by a single systematic risk factor Y. Default indicators are mutually independent conditional on Y.**
The ASRF Model

Theorem 1.1

Let us denote the exposure share of obligor \( n \) by \( s_n = \frac{E_n}{\sum_{n=1}^{N} E_n} \). Then, under assumption 1.3 the portfolio loss ratio \( \mathcal{L} = \sum_{n=1}^{N} s_n \cdot L_n \cdot D_n \) conditional on any realization \( y \) of the systematic risk factor \( Y \) satisfies,

\[
\mathcal{L} - \mathbb{E}(\mathcal{L} | Y = y) \to 0 \text{ almost surely as } N \to \infty.
\]

Under one-factor Merton model and assuming \( L_n \) to be deterministic,

\[
\mathbb{E}(\mathcal{L} | Y = y) = \sum_{n=1}^{N} s_n \cdot L_n \cdot \Phi \left( \frac{t_n - \sqrt{\rho_n} y}{\sqrt{1 - \rho_n}} \right). \tag{5}
\]
By Theorem 1.1,

\[ \text{VaR}_\alpha(\mathcal{L}) - \mathbb{E}(\mathcal{L}|Y = l_{1-\alpha}(Y)) \to 0 \text{ a.s. as } N \to \infty. \]

Finally,

\[ \text{VaR}^A_\alpha = \sum_{n=1}^{N} s_n \cdot L_n \cdot \Phi \left( \frac{t_n + \sqrt{\rho_n} \Phi^{-1}(\alpha)}{\sqrt{1 - \rho_n}} \right), \]

and,

\[ \text{VaRC}^A_{\alpha,n} = s_n \cdot \frac{\partial \text{VaR}^A_\alpha}{\partial s_n} = s_n \cdot L_n \cdot \Phi \left( \frac{t_n + \sqrt{\rho_n} \Phi^{-1}(\alpha)}{\sqrt{1 - \rho_n}} \right). \]
Concentration Risk

However:

- Real world portfolios are not perfectly fine-grained.
- The ASRF model might be approximately valid for huge portfolios but less satisfactory for portfolios of smaller institutions (or more specialized).
- The formula can underestimate the required economic capital.
- Does not allow the measurement of sector concentration risk.

In practice: Monte Carlo simulations (robust but computationally intensive). The variance is an issue.

Proposal: A new method based on wavelets to overcome the computational complexity.
Outline

1. Portfolio Credit Risk Modeling
2. Haar Wavelets for Laplace Transform Inversion
3. The WA Method to Quantify Losses
4. Credit Risk Contributions
5. The WA Extension to the Multi-Factor Model
6. Conclusions
Consider \( f \in L^2(\mathbb{R}) \).

\[
V_j = \{ g \in L^2 : g \text{ is constant on } l_{j,k}, \ k \in \mathbb{Z} \} \text{ and } l_{j,k} = \left[ \frac{k}{2^j}, \frac{k+1}{2^j} \right).
\]

An orthogonal basis for \( V_j \) is given by the family:

\[
\phi_{j,k}(x) = 2^j \phi(2^j x - k), \ k \in \mathbb{Z}, \ \phi(x) = \chi_{[0,1)}(x).
\]

Consider the orthogonal projector onto \( V_j \), \( \mathcal{P}_j : L^2(\mathbb{R}) \rightarrow V_j \).

We can write,

\[
\mathcal{P}_j f = \sum_{k=-\infty}^{\infty} c_{j,k} \phi_{j,k},
\]

\[
c_{j,k} = \langle f, \phi_{j,k} \rangle = 2^j \int_{l_{j,k}} f(x)dx.
\]
Example: component $P_4 f$ on $[0, 1]$, for $f(x) = \sin(2\pi x)$. 
\( \phi \) is called the **scaling function** (approximation at certain level)

**Figure**: Scaling \((\phi_{2,3})\) function.
The Haar System

**Example:** the step function

\[
f(x) = \begin{cases} 
-1, & x \in [-\frac{1}{2}, 0], \\
1, & x \in (0, \frac{1}{2}], \\
0, & \text{otherwise}.
\end{cases}
\]

This function is poorly approximated by its Fourier series:

Wavelets are more flexible:

\[
f(x) = \frac{\sqrt{2}}{2} \phi_{1,0}(x) - \frac{\sqrt{2}}{2} \phi_{1,-1}(x).
\]
Laplace Transform

Define the **Laplace transform** of \( f \),

\[
\tilde{f}(s) = \int_{0}^{+\infty} e^{-sx} f(x) \, dx = \lim_{\tau \to +\infty} \int_{\tau}^{T} e^{-sx} f(x) \, dx, \quad s \in \mathbb{C}.
\]

**Theorem 2.1**

*(Bromwich inversion integral)* If the Laplace transform of \( f(x) \) exists, then,

\[
f(x) = \lim_{k \to +\infty} \frac{1}{2\pi i} \int_{\sigma - ik}^{\sigma + ik} \tilde{f}(s) e^{sx} \, ds, \quad x > 0,
\]

where \(|f(x)| \leq e^{\Sigma x}\) for some positive real number \( \Sigma \) and \( \sigma \) is any other real number such that \( \sigma > \Sigma \).
Let $f$ be a function in $L^2([0, 1])$.

$$f(x) = \lim_{m \to +\infty} f_m(x), \quad f_m(x) = \sum_{k=0}^{2^m-1} c_{m,k} \phi_{m,k}(x),$$

where,

$$c_{m,k} = \int_{k \frac{2^m}{k+1}}^{k+1 \frac{2^m}{k}} f(x) \phi_{m,k}(x) dx,$$

$k = 0, \ldots, 2^m - 1.$
Consider the Laplace Transform of $f$:
\[
\tilde{f}(s) = \int_0^{+\infty} e^{-sx} f(x) \, dx, \quad \text{(assume } f(x) = 0, \forall x \notin [0, 1]).
\]

Wavelet Approximation (WA) method: approximate $\tilde{f}$ by $\tilde{f}_m$ and compute the coefficients $c_{m,k}$.

\[
\tilde{f}(s) = \int_0^{+\infty} e^{-sx} f(x) \, dx \simeq \int_0^{+\infty} e^{-sx} f_m(x) \, dx = \\
\frac{2^m}{s} \left( 1 - e^{-s \frac{1}{2^m}} \right) \sum_{k=0}^{2^m-1} c_{m,k} e^{-s \frac{k}{2^m}}.
\]

Change of variable $z = e^{-s \frac{1}{2^m}}$: \[
\sum_{k=0}^{2^m-1} c_{m,k} z^k \simeq Q_m(z).
\]
The WA Method

We obtain the coefficients $c_{m,k}$ by means of the Cauchy’s integral formula,

$$c_{m,k} \simeq \frac{2}{\pi r^k} \int_0^\pi \Re(Q_m(re^{i\mu})) \cos(ku) du, \quad k = 0, \ldots, 2^m - 1.$$

The integral can be evaluated by means of the trapezoidal rule,

$$I(r, k) = \int_0^\pi \Re(Q_m(re^{i\mu})) \cos(ku) du,$$

$$I(r, k; h) = \frac{h}{2} \left( Q_m(r) + (-1)^k Q_m(-r) + 2 \sum_{j=1}^{m_T-1} \Re(Q_m(re^{ih_j})) \cos(kh_j) \right),$$

where $h = \frac{\pi}{m_T}$ and $h_j = jh$ for all $j = 0, \ldots, m_T$.

Then,

$$c_{m,k} \simeq \frac{2}{\pi r^k} I(r, k) \simeq \frac{2}{\pi r^k} I(r, k; h), \quad k = 1, \ldots, 2^m - 1.$$
Outline

1. Portfolio Credit Risk Modeling
2. Haar Wavelets for Laplace Transform Inversion
3. The WA Method to Quantify Losses
4. Credit Risk Contributions
5. The WA Extension to the Multi-Factor Model
6. Conclusions
Focus on the **one-factor Merton** model. **Assume:**

\[ L_n = 100\%, \sum_{n=1}^{N} E_n = 1. \]

Let \( F \) be the CDF of \( L \) and \( f_L \) its PDF.

\[ r_n = \sqrt{\rho} Y + \sqrt{1 - \rho} \epsilon_n, \]

\( (Y, \epsilon_n \text{ i.i.d. } N(0, 1)) \).

Conditional default probabilities, \( P_n(y) \equiv \Phi \left( \frac{t_n - \sqrt{\rho} y}{\sqrt{1 - \rho}} \right), t_n = \Phi^{-1}(P_n). \)
Consider,

\[ F(x) = \begin{cases} \bar{F}(x), & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } x > 1, \end{cases} \]

Define unconditional MGF: \( \tilde{M}_L(s) \equiv \mathbb{E}(e^{-sL}) \).

### Assumption 3.1

**Conditional Independence Framework.** *If the systematic factor \( Y \) is fixed, defaults occur independently because the only remaining uncertainty is the idiosyncratic risk.*

Define conditional MGF:

\[ \tilde{M}_L(s; y) \equiv \mathbb{E}(e^{-sL} \mid Y = y) = \prod_{n=1}^{N} \left[ 1 - P_n(y) + P_n(y)e^{-sE_n} \right]. \]

Then:

\[ \tilde{M}_L(s) = \mathbb{E}(\tilde{M}_L(s; y)) = \int_{\mathbb{R}} \prod_{n=1}^{N} \left[ 1 - P_n(y) + P_n(y)e^{-sE_n} \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy. \]
Since: $\bar{F} \in L^2([0, 1])$ then:

$$\bar{F}(x) \simeq F_m(x), \quad F_m(x) = \sum_{k=0}^{2^m-1} c_{m,k} \phi_{m,k}(x),$$

$$\bar{F}(x) = \lim_{m \to +\infty} F_m(x).$$

Observe:

$$\tilde{M}_L(s) = \int_0^{+\infty} e^{-sx} F'(x) dx = e^{-s} + s \int_0^1 e^{-sx} \bar{F}(x) dx.$$  

Then: $\left( \tilde{M}_L(s) - e^{-s} \right) / s$ is the Laplace transform of $\bar{F}$.

Apply the WA method,

Compute: $c_{m,k}$. 

We have: $\bar{F}(\text{VaR}_\alpha) \simeq \bar{F}_m(\text{VaR}_\alpha) = 2^{\frac{m}{2}} \cdot c_m, \bar{k}, \bar{k} \in \{0, 1, \ldots, 2^m - 1\}$.

**Algorithm:**

1. Compute $\bar{F}_m(\frac{2^m - 1}{2^m}) = 2^{\frac{m}{2}} \cdot c_m, 2^m - 1$.
2. If $\bar{F}_m(\frac{2^m - 1}{2^m}) > \alpha$ then compute $\bar{F}_m(\frac{2^m - 1 - 2^{m-2}}{2^m})$, otherwise compute $\bar{F}_m(\frac{2^m - 1 + 2^{m-2}}{2^m})$.
3. Finish after $m$ steps storing the $\bar{k}$ value s.t. $\bar{F}_m(\frac{\bar{k}}{2^m})$ is the closest value to $\alpha$.

In fact, $\bar{F}_m(\xi) = \bar{F}_m(\frac{\bar{k}}{2^m})$, for all $\xi \in \left[\frac{\bar{k}}{2^m}, \frac{\bar{k}+1}{2^m}\right]$.

So take: $\text{VaR}^{W(m)} = \frac{2\bar{k}+1}{2^{m+1}}$. 
Parameters: \( m = 10, m_T = 2^m, l = 20 \). MC with \( 5 \times 10^6 \) scenarios.

Portfolio 3.1

*We consider* \( N = 102 \) obligors, with \( P_n = 0.1\%, \ E_n = 1, n = 1, \ldots, 100, \ E_{101} = E_{102} = 20, \ \rho = 0.3 \) *and* \( L_n = 1 \).

<table>
<thead>
<tr>
<th>Method</th>
<th>( \text{VaR}_{0.999} )</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monte Carlo</td>
<td>0.1500</td>
<td></td>
</tr>
<tr>
<td>ASRF</td>
<td>0.0474</td>
<td>(-68.39%)</td>
</tr>
<tr>
<td>Saddle Point</td>
<td>0.1270</td>
<td>(-15.37%)</td>
</tr>
<tr>
<td>Wavelet Approximation</td>
<td>0.1490</td>
<td>(-0.69%)</td>
</tr>
</tbody>
</table>
Figure: Tail probability approximation of a heterogeneous portfolio with severe name concentration.
<table>
<thead>
<tr>
<th>Portfolio</th>
<th>$N$</th>
<th>$P_n$</th>
<th>$E_n$</th>
<th>$\rho$</th>
<th>HHI</th>
<th>$\frac{1}{N}$</th>
</tr>
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<tbody>
<tr>
<td>P1</td>
<td>100</td>
<td>0.21%</td>
<td>$\frac{C}{n}$</td>
<td>0.15</td>
<td>0.0608</td>
<td>0.0100</td>
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<tr>
<td>P2</td>
<td>1000</td>
<td>1.00%</td>
<td>$\frac{C}{n}$</td>
<td>0.15</td>
<td>0.0293</td>
<td>0.0010</td>
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<tr>
<td>P3</td>
<td>1000</td>
<td>0.30%</td>
<td>$\frac{C}{n}$</td>
<td>0.15</td>
<td>0.0293</td>
<td>0.0010</td>
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<tr>
<td>P4</td>
<td>10000</td>
<td>1.00%</td>
<td>$\frac{C}{n}$</td>
<td>0.15</td>
<td>0.0172</td>
<td>0.0001</td>
</tr>
<tr>
<td>P5</td>
<td>20</td>
<td>1.00%</td>
<td>$\frac{1}{N}$</td>
<td>0.5</td>
<td>0.0500</td>
<td>0.0500</td>
</tr>
<tr>
<td>P6</td>
<td>10</td>
<td>0.21%</td>
<td>$\frac{C}{n}$</td>
<td>0.5</td>
<td>0.1806</td>
<td>0.1000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>$\text{VaR}^{W(8)}_{0.999}$</th>
<th>$\text{RE}(0.999, 8)$</th>
<th>$\text{VaR}^{W(9)}_{0.999}$</th>
<th>$\text{RE}(0.999, 9)$</th>
<th>$\text{VaR}^{W(10)}_{0.999}$</th>
<th>$\text{RE}(0.999, 10)$</th>
<th>$\text{VaR}^{M}_{0.999}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>0.1934</td>
<td>-0.19%</td>
<td>0.1963</td>
<td>1.32%</td>
<td>0.1938</td>
<td>0.06%</td>
<td>0.1937</td>
</tr>
<tr>
<td>P2</td>
<td>0.1934</td>
<td>1.01%</td>
<td>0.1924</td>
<td>0.50%</td>
<td>0.1919</td>
<td>0.25%</td>
<td>0.1914</td>
</tr>
<tr>
<td>P3</td>
<td>0.1426</td>
<td>1.46%</td>
<td>0.1416</td>
<td>0.77%</td>
<td>0.1411</td>
<td>0.42%</td>
<td>0.1405</td>
</tr>
<tr>
<td>P4</td>
<td>0.1621</td>
<td>0.24%</td>
<td>0.1611</td>
<td>-0.36%</td>
<td>0.1616</td>
<td>-0.06%</td>
<td>0.1617</td>
</tr>
</tbody>
</table>
Portfolio Credit Risk Modeling
Haar Wavelets for Laplace Transform Inversion
The WA Method to Quantify Losses
Credit Risk Contributions
The WA Extension to the Multi-Factor Model
Conclusions

VaR Computation with the WA Method
Numerical Examples

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>$\text{VaR}^{W(8)}_{0.999}$</th>
<th>$\text{VaR}^{W(9)}_{0.999}$</th>
<th>$\text{VaR}^{W(10)}_{0.999}$</th>
<th>$\text{VaR}^{M}_{0.999}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>0.2</td>
<td>0.4</td>
<td>0.7</td>
<td>58.3</td>
</tr>
<tr>
<td>P2</td>
<td>1.8</td>
<td>3.6</td>
<td>7.2</td>
<td>571.6</td>
</tr>
<tr>
<td>P3</td>
<td>1.8</td>
<td>3.6</td>
<td>7.2</td>
<td>567.6</td>
</tr>
<tr>
<td>P4</td>
<td>18.2</td>
<td>36.1</td>
<td>72.4</td>
<td>1379.1</td>
</tr>
</tbody>
</table>

**Table:** CPU time (in seconds).

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>$\text{VaR}^{W(10)}_{0.9999}$</th>
<th>$\text{RE}(0.9999, 10)$</th>
<th>$\text{VaR}^{W(10)}_{0.99999}$</th>
<th>$\text{RE}(0.99999, 10)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>0.2251</td>
<td>−0.07%</td>
<td>0.2935</td>
<td>−1.70%</td>
</tr>
<tr>
<td>P2</td>
<td>0.2622</td>
<td>−0.46%</td>
<td>0.3325</td>
<td>−1.80%</td>
</tr>
<tr>
<td>P3</td>
<td>0.1812</td>
<td>−0.10%</td>
<td>0.2290</td>
<td>−1.88%</td>
</tr>
<tr>
<td>P4</td>
<td>0.2261</td>
<td>−0.25%</td>
<td>0.2935</td>
<td>−1.30%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>$\text{VaR}^{W(10)}_{0.9999}$</th>
<th>$\text{RE}(0.99999, 10)$</th>
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<th>$\text{RE}(0.99999, 10)$</th>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>$\text{VaR}^{M}_{0.9999}$</th>
<th>$\text{VaR}^{M}_{0.99999}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>0.2253</td>
<td>0.2985</td>
</tr>
<tr>
<td>P2</td>
<td>0.2634</td>
<td>0.3386</td>
</tr>
<tr>
<td>P3</td>
<td>0.1813</td>
<td>0.2334</td>
</tr>
<tr>
<td>P4</td>
<td>0.2267</td>
<td>0.2973</td>
</tr>
</tbody>
</table>
Figure: Tail probability approximation of Portfolios P1 at scale $m = 10$. 
**Figure:** Tail probability approximation of Portfolios P2 at scale $m = 10$. 

The graph illustrates the comparison between Wavelet Approximation (scale 10), Monte Carlo, and ASRF methods for tail probability approximation.
Figure: Tail probability approximation of Portfolios P3 at scale $m = 10$. 
Figure: Tail probability approximation of Portfolios P4 at scale $m = 10$. 
Outline

1. Portfolio Credit Risk Modeling
2. Haar Wavelets for Laplace Transform Inversion
3. The WA Method to Quantify Losses
4. Credit Risk Contributions
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6. Conclusions
The Expected Shortfall

By definition: 

\[ \text{ES}_\alpha(E) = \frac{1}{1-\alpha} \int_{\text{VaR}_\alpha(E)}^{+\infty} xf_L(E, x) \, dx. \]

Integrating by parts,

\[ \text{ES}_\alpha(E) = \frac{1}{1-\alpha} \left( 1 - \alpha \text{VaR}_\alpha(E) - \int_{\text{VaR}_\alpha(E)}^{1} \bar{F}(E, x) \, dx \right) \approx \text{ES}^{W(m)}_\alpha(E), \]

where,

\[ \text{ES}^{W(m)}_\alpha(E) \equiv \frac{1}{1-\alpha} \left( 1 - \alpha \text{VaR}^{W(m)}_\alpha(E) - \frac{1}{2^m+1} c_{m, \bar{k}}(E) - \frac{1}{2^m} \sum_{k={\bar{k}+1}}^{2^m-1} c_{m, k}(E) \right) \]
Taking into account that $\bar{F}(E, \text{VaR}_\alpha) = \alpha$ then,

$$\text{VaRC}_{\alpha,i} \equiv E_i \cdot \frac{\partial \text{VaR}_\alpha}{\partial E_i} = -E_i \cdot \frac{\partial \bar{F}(E, \text{VaR}_\alpha)}{\partial E_i} = \frac{-E_i \cdot \partial \bar{F}(E, \text{VaR}_\alpha)}{f_L(E, \text{VaR}_\alpha)}. \quad (6)$$

Recall that: $\bar{F}(E, x) \simeq \sum_{k=0}^{2^m-1} c_{m,k}(E) \phi_{m,k}(x)$.

**Numerator:** $\frac{\partial \bar{F}(E,x)}{\partial E_i} \simeq \sum_{k=0}^{2^m-1} \frac{\partial c_{m,k}(E)}{\partial E_i} \phi_{m,k}(x)$.

Evaluating this expression in $\text{VaR}_\alpha$:

$$\frac{\partial \bar{F}(E, \text{VaR}_\alpha)}{\partial E_i} \simeq 2^m \frac{\partial c_{m,k}(E)}{\partial E_i}.$$  

Finally, $\text{VaRC}_{\alpha,i} \simeq \text{VaRC}_{\alpha,i}^{W(m)}$, where $\text{VaRC}_{\alpha,i}^{W(m)} \equiv C \cdot E_i \cdot \frac{\partial c_{m,k}(E)}{\partial E_i}$ and $C$ is a constant such that $\sum_{i=1}^{N} \text{VaRC}_{\alpha,i}^{W(m)} = \text{VaR}_{\alpha}^{W(m)}$.  

ES Contributions

Take partial derivatives w.r.t. the exposures,

\[ \text{ESC}_{\alpha,i} \equiv E_i \cdot \frac{\partial \text{ES}_\alpha}{\partial E_i} \]

\[ = E_i \cdot \frac{1}{1 - \alpha} \left( -\alpha \frac{\partial \text{VaR}_\alpha}{\partial E_i} + \frac{\partial \text{VaR}_\alpha}{\partial E_i} \overline{F}(E, \text{VaR}_\alpha) - \int_{\text{VaR}_\alpha}^{1} \frac{\partial \overline{F}(E, x)}{\partial E_i} dx \right) \]

\[ \simeq \text{ESC}_{\alpha,i}^{W(m)}, \]

where,

\[ \text{ESC}_{\alpha,i}^{W(m)} \equiv -E_i \cdot \frac{1}{2^m} \cdot \frac{1}{1 - \alpha} \cdot \left( \frac{1}{2} \frac{\partial c_{m,k}(E)}{\partial E_i} + \sum_{k=k+1}^{2^m-1} \frac{\partial c_{m,k}(E)}{\partial E_i} \right). \]
Parameters: \( m = 10, m_T = 2^m \). Contributions: MC with \( 10^8 \) scenarios.

Portfolio 4.1

*This portfolio has* \( N = 10000 \) obligors with \( \rho = 0.15, P_n = 0.01 \) and \( E_n = \frac{1}{n} \) for \( n = 1, \ldots, N \), as Portfolio P4.

<table>
<thead>
<tr>
<th>Method</th>
<th>( l/2 - n_2 )</th>
<th>( l/2 - n_1 )</th>
<th>( \alpha = 0.9999 )</th>
<th>( \alpha = 0.99999 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{VaR}_\alpha^M )</td>
<td></td>
<td></td>
<td>0.2267</td>
<td>0.2973</td>
</tr>
<tr>
<td>( \text{VaR}_\alpha^A )</td>
<td></td>
<td></td>
<td>0.1683 (−25.76%)</td>
<td>0.2322 (−21.91%)</td>
</tr>
<tr>
<td>( \text{VaR}_\alpha^W(10) ) (20)</td>
<td>10</td>
<td>10</td>
<td>0.2261 (−0.25%)</td>
<td>0.2935 (−1.30%)</td>
</tr>
<tr>
<td>( \text{VaR}_\alpha^W(10) ) (20, ( 4 \cdot 10^{-1} ))</td>
<td>9</td>
<td>0</td>
<td>0.2261 (−0.25%)</td>
<td>0.2944 (−0.97%)</td>
</tr>
<tr>
<td>( \text{VaR}_\alpha^W(10) ) (20, ( 6 \cdot 10^{-1} ))</td>
<td>7</td>
<td>0</td>
<td>0.2261 (−0.25%)</td>
<td>0.2944 (−0.97%)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Method</th>
<th>( l/2 - n_2 )</th>
<th>( l/2 - n_1 )</th>
<th>( \alpha = 0.99 )</th>
<th>( \alpha = 0.999 )</th>
<th>( \alpha = 0.9999 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{ES}_\alpha^M )</td>
<td></td>
<td></td>
<td>0.1290</td>
<td>0.1895</td>
<td>0.2553</td>
</tr>
<tr>
<td>( \text{ES}_\alpha^W(10) ) (20)</td>
<td>10</td>
<td>10</td>
<td>0.1290 (−0.02%)</td>
<td>0.1895 (−0.01%)</td>
<td>0.2556 (0.12%)</td>
</tr>
<tr>
<td>( \text{ES}_\alpha^W(10) ) (20, ( 4 \cdot 10^{-1} ))</td>
<td>9</td>
<td>0</td>
<td>0.1289 (−0.12%)</td>
<td>0.1895 (−0.01%)</td>
<td>0.2556 (0.12%)</td>
</tr>
<tr>
<td>( \text{ES}_\alpha^W(10) ) (20, ( 6 \cdot 10^{-1} ))</td>
<td>7</td>
<td>0</td>
<td>0.1289 (−0.11%)</td>
<td>0.1896 (0.00%)</td>
<td>0.2559 (0.25%)</td>
</tr>
</tbody>
</table>
**Figure:** The 250 biggest risk contributions to the ES at 99% confidence level for the Portfolio 4.1.
Figure: The 250 smallest risk contributions to the ES at 99% confidence level for the Portfolio 4.1.
Figure: The 250 biggest risk contributions to the ES at 99.9% confidence level for the Portfolio 4.1.
Figure: The 250 smallest risk contributions to the ES at 99.9% confidence level for the Portfolio 4.1.
### The Expected Shortfall and the Risk Contributions

#### Numerical Examples

<table>
<thead>
<tr>
<th>Method</th>
<th>$l/2 - \bar{n}_2$</th>
<th>$l/2 - \bar{n}_1$</th>
<th>$\alpha = 0.99$</th>
<th>$\alpha = 0.999$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{n=1}^{N} \text{ESC}_{\alpha,n}^M$</td>
<td></td>
<td>0.1290</td>
<td>0.1892</td>
<td></td>
</tr>
<tr>
<td>$\sum_{n=1}^{N} \text{ESC}_{\alpha,n}^{W(10)}(20)$</td>
<td>10</td>
<td>10</td>
<td>0.1293</td>
<td>0.1891</td>
</tr>
<tr>
<td>$\sum_{n=1}^{N} \text{ESC}_{\alpha,n}^{W(10)}(20, 10^{-4})$</td>
<td>10</td>
<td>4</td>
<td>0.1293</td>
<td>0.1890</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Method</th>
<th>CPU time (in seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{VaR}_{\alpha}^{W(10)}(20, 6 \cdot 10^{-1})$</td>
<td>25.3</td>
</tr>
<tr>
<td>$\text{VaR}_{\alpha}^{W(9)}(20, 6 \cdot 10^{-1})$</td>
<td>12.7</td>
</tr>
<tr>
<td>$\text{VaR}_{\alpha}^{W(8)}(20, 6 \cdot 10^{-1})$</td>
<td>6.4</td>
</tr>
<tr>
<td>$\text{ESC}_{\alpha,n}^{W(10)}(20, 10^{-4})$</td>
<td>622.5</td>
</tr>
</tbody>
</table>
Portfolio 4.2

This portfolio has \( N = 1001 \) obligors, with \( E_n = 1 \) for \( n = 1, \ldots, 1000 \), and one obligor with \( E_{1001} = 100 \). \( P_n = 0.0033 \) for all the obligors and \( \rho = 0.2 \).

<table>
<thead>
<tr>
<th>Method</th>
<th>CPU time (in seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{VaR}_{\alpha}^{W(10)}(64, 5 \cdot 10^{-1}) )</td>
<td>5.6</td>
</tr>
<tr>
<td>( \text{ESC}_{\alpha,n}^{W(10)}(64, 10^{-4}) )</td>
<td>78.4</td>
</tr>
</tbody>
</table>
Figure: Tail probability approximation for Portfolio 4.2.
This portfolio has $N = 100$ obligors, all them with $P_n = 0.01$, $\rho = 0.5$ and exposures (as in Glasserman (2005)),

$$E_n = \begin{cases} 1, & n = 1, \ldots, 20, \\ 4, & n = 21, \ldots, 40, \\ 9, & n = 41, \ldots, 60, \\ 16, & n = 61, \ldots, 80, \\ 25, & n = 81, \ldots, 100, \end{cases}$$
Figure: VaR contributions at 99.9% confidence level for Portfolio 4.3 using the WA method and GH integration formulas with 64 nodes and $\bar{c} = 10^{-4}$. 
**Figure:** Expected Shortfall contributions at 99.9% confidence level for Portfolio 4.3 using the WA method and GH integration formulas with 64 nodes and $\bar{\epsilon} = 10^{-4}$. 
Figure: Expected Shortfall contributions at 99.99% confidence level for Portfolio 4.3 using the WA method and GH integration formulas with 64 nodes and $\bar{\epsilon} = 10^{-4}$. 
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Multi-factor Gaussian copula:

$$W_n = a_n^T Y + b_n Z_n.$$  \(7\)

Multi-factor t-copula:

$$W_n = \sqrt{\frac{\nu}{V}} \left( a_n^T Y + b_n Z_n \right).$$  \(8\)

The computation of characteristic functions (or Laplace transforms) are not affordable by numerical quadrature.
Outline

1. Portfolio Credit Risk Modeling
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Conclusions

- New method for Laplace Transform inversion based on Haar wavelets. Particularly well suited for stepped-shape functions, where other state-of-the-art methods fail.
- Computation of the VaR and the ES risk measures under the one-factor Merton model. Very accurate and fast, even in the presence of severe name concentration. \( \text{Rel. err.} < 1\% \).
- The WA method computes the entire distribution of losses without extra computational time (CDO pricing).
- Accurately computation of the risk contributions to the VaR and the ES. \( \text{Rel. err.} < 1\% \).
- Extension to the multi-factor Merton model (to account for sector concentration).
References

