Name concentration risk and its measurement

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Outline

Introduction

- It is very important for a bank to manage the risks originated from its business activities. The credit risk underlying the credit portfolio is often the largest risk in a bank.
- Basel Accords laid the basis for international minimum capital standards. Banks became subject to **regulatory capital** requirements.
- Basel II is structured in a three Pillar framework:
  - Pillar 1: more risk sensitive minimal capital requirements.
  - Pillar 2: banks are allowed to calculate the **economic capital** (**risk concentration**).
  - Pillar 3: transparency in bank's financial reporting.

Introduction

- Concentration risks arise from an unequal distribution of loans to single borrowers (**exposure or name concentration**) or different industry or regional sectors (**sector concentration**).
- Within Basell II banks may opt for the **standard approach** (more conservative) or the **internal rating based** (IRB) approach (more advanced).
- Merton model: basis of the Basel II IRB approach. Under homogeneity conditions, this model leads to the **ASRF** model. However, this model can **underestimate risks** in the presence of exposure concentration.
- Credit risk managers are interested in:
  - How can concentration risk be quantified?
  - How can risk measures be accurately computed in short times?
  - How does the individual transactions contribute to the total risk?

Risk Parameters

- We specify a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. We fix a time horizon $T > 0$ (usually one year).
- We consider a credit portfolio consisting of N obligors.
- Any obligor n is characterized by:
    - The **exposure at default** E_n: potential exposure measured in currency.
    - The **loss given default** L_n: magnitude of likely loss on the exposure as a percentage of the exposure.
    - The **probability of default** P_n: likelihood that a loan will not be repaid.

Each of them can be estimated from empirical default data.

Risk Measures

Consider an obligor n subject to default in the fixed time horizon T.

We introduce D_n, the default indicator of obligor n,

$$D_n = \begin{cases} 1, & \text{if obligor } n \text{ is in default,} \\ 0, & \text{if obligor } n \text{ is not in default,} \end{cases}$$

where $\mathbb{P}(D_n = 1) = P_n$ and $\mathbb{P}(D_n = 0) = 1 - P_n$.

Let $\mathcal{L}$ be the portfolio loss given by,

$$\mathcal{L} = \sum_{n=1}^{N} \mathcal{L}_n,$$

where $\mathcal{L}_n = E_n \cdot L_n \cdot D_n$.

Risk Measures

Credit risk can split in **Expected Losses** EL (which can be forecasted) and **Unexpected Losses** UL (more difficult to quantify).

Assumption 1.1

The exposure at default E_n, the loss given default L_n and the default indicator D_n of an obligor n are independent.

Denote by EL_n the expectation value of L_n, therefore,

$$\mathrm{EL} = \mathbb{E}(\mathcal{L}) = \sum_{n=1}^{N} E_n \cdot EL_n \cdot P_n.$$

Holding the $\mathrm{UL} = \sqrt{\mathbb{V}(\mathcal{L})}$ as a risk capital for cases of financial distress might not be appropriate (peak losses can be very large when they occur).

Risk Measures

Let $\alpha \in (0,1)$ be a given confidence level, the α-quantile of the loss distribution of $\mathcal{L}$ in this context is called **Value at Risk** (VaR). Thus,

$$\text{VaR}_\alpha = \inf\{l \in \mathbb{R} : \mathbb{P}(\mathcal{L} \leq l) \geq \alpha\} = \inf\{l \in \mathbb{R} : F_{\mathcal{L}}(l) \geq \alpha\},$$

where $F_{\mathcal{L}}$ is the cumulative distribution function of the loss variable $\mathcal{L}$.

VaR is the measure chosen in the Basel II Accord ($\alpha = 0.999$) for the computation of capital requirement.

Another important risk measure is the so called economic capital EC_α for a given confidence level α,

$$\text{EC}_\alpha = \text{VaR}_\alpha - \text{EL}.$$

Risk Measures: VaR

Drawbacks:

1. VaR gives no information about the severity of losses which occur with probability less than $1 - \alpha$.
2. VaR is not a coherent risk measure, since it is not sub-additive. If $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ then $\text{VaR}_\alpha(\mathcal{L}) \nleq \text{VaR}_\alpha(\mathcal{L}_1) + \text{VaR}_\alpha(\mathcal{L}_2)$ (this fact contradicts the intuition of diversification benefits associated with merging portfolios).

Example

Consider two independent loans with default indicators following a Bernoulli distribution $B(1, p)$ with $0.006 \leq p < 0.01$ and exposures equal to 1. Define two portfolios A and B, each of them consisting of one unit of the above introduced loans. Then if we denote the corresponding portfolio losses by $\mathcal{L}_A$ and $\mathcal{L}_B$,

$$\mathrm{VaR}_{0.99}(\mathcal{L}_A) = \mathrm{VaR}_{0.99}(\mathcal{L}_B) = 0.$$

Now if we consider a portfolio C defined as the union of portfolios A and B and denote by $\mathcal{L}_C = \mathcal{L}_A + \mathcal{L}_B$. Then,

$$\mathbb{P}(\mathcal{L}_C = 0) = (1 - p)^2 < 0.99,$$

and therefore,

$$\mathrm{VaR}_{0.99}(\mathcal{L}_C) > 0,$$

so that,

$$\mathrm{VaR}_{0.99}(\mathcal{L}_C) > \mathrm{VaR}_{0.99}(\mathcal{L}_A) + \mathrm{VaR}_{0.99}(\mathcal{L}_B).$$

Risk Measures: Expected Shortfall

On the contrary the **Expected Shortfall** (ES) enjoys the coherence properties.

$$\mathrm{ES}_\alpha = \mathbb{E}\left(\mathcal{L} | \mathcal{L} \geq \mathrm{VaR}_\alpha\right),$$

or alternatively,

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha} \int_{VaR_\alpha}^{+\infty} x f_{\mathcal{L}}(x) dx,$$

($f_{\mathcal{L}}$ density function of the loss variable $\mathcal{L}$).

Risk Contributions

To decompose the risk measured by the VaR or the ES into individual transactions (allocation principle, **D. Tasche (2000)**) define the **Risk Contribution to VaR** (VaRC) of obligor n at confidence level α by,

$$\mathrm{VaRC}_{\alpha,n} \equiv E_n \cdot \frac{\partial \mathrm{VaR}_\alpha}{\partial E_n},$$

and the **Risk Contribution to ES** (ESC) of obligor n at confidence level α by,

$$\mathrm{ESC}_{\alpha,n} \equiv E_n \cdot \frac{\partial \mathrm{ES}_\alpha}{\partial E_n}.$$

These definitions satisfy the additivity condition,

$$\sum_{n=1}^{N} \mathrm{VaRC}_{\alpha,n} = \mathrm{VaR}_\alpha, \quad \sum_{n=1}^{N} \mathrm{ESC}_{\alpha,n} = \mathrm{ES}_\alpha.$$

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Risk Contributions

Under appropriate conditions the marginal VaR contribution at confidence level α of the obligor n is,

$$\mathrm{VaRC}_{\alpha,n} = \mathbb{E}(\mathcal{L}_n | \mathcal{L} = \mathrm{VaR}_\alpha), \tag{1}$$

and the marginal contribution at confidence level α to the expected shortfall is,

$$\mathrm{ESC}_{\alpha,n} = \mathbb{E}(\mathcal{L}_n | \mathcal{L} \geq \mathrm{VaR}_\alpha). \tag{2}$$

General Framework

- Credit risk models can be divided into two fundamental classes of models, **structural** or **asset-value models** and **reduced-form** or **default-rate models**.
- Asset-value models have their origin on the famous **Merton** model, where the default of a firm is modeled in terms of the relationship between its assets and the liabilities that it faces at the end of a given time period.
- Two famous industry models descending from the Merton approach are the **KMV** model (developed by Moody's KMV) and the **CreditMetrics** model (developed by JPMorgan and the RiskMetrics Group).

General Framework

- Company's debt is given by a zero-coupon bond with face value B.
- The model also assumes that the asset value process $(V_t)_{t\geq 0}$ follows a geometric Brownian motion of the form,

$$dV_t = \mu_V V_t dt + \sigma_V V_t dW_t, \quad 0 \leq t \leq T. \tag{3}$$

- The solution at time T of the stochastic differential equation (3) with initial value V_0 can be computed and is given by,

$$V_T = V_0 e^{\left(\mu_V - \frac{1}{2}\sigma_V^2\right)T + \sigma_V W_T}.$$

- This implies in particular that,

$$\log V_T \sim N\left(\log V_0 + \left(\mu_V - \frac{1}{2}\sigma_V^2\right)T, \sigma_V^2 T\right).$$

- The default probability of the firm by time T can be computed as,

$$\mathbb{P}(V_T \leq B) = \mathbb{P}(\log V_T \leq \log B) = \Phi\left(\frac{\log \frac{B}{V_0} - (\mu_V - \frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}\right).$$

The Multi-Factor Merton Model

We define $V_t^{(n)}$ to be the asset value of the counterparty n at time $t \leq T$.
For every counterparty $\exists T_n$ s.t. counterparty n defaults in the time period $[0, T]$ if $V_T^{(n)} < T_n$.
For $n = 1, \ldots, N$, we define,

$$D_n = \chi_{\{V_T^{(n)} < T_n\}} \sim B\left(1, \mathbb{P}(V_T^{(n)} < T_n)\right). \quad (4)$$

Consider the borrower n's asset-value log return r_n: $\log\left(V_T^{(n)}/V_0^{(n)}\right)$.

Assumption 1.2

Asset returns r_n depend linearly on K standard normally distributed risk factors $X = (X_1, \ldots, X_K)$ affecting the borrowers' defaults in a systematic way as well as on a standard normally distributed idiosyncratic term ϵ_n. Moreover, ϵ_n are independent of the systematic factors X_k for every $k \in \{1, \ldots, K\}$ and the ϵ_n are uncorrelated.

The Multi-Factor Merton Model

Under this assumption and after standardization,

$$r_n = \beta_n Y_n + \sqrt{1-\beta_n^2}\epsilon_n.$$

Y_n can be decomposed into K independent factors $X = (X_1, \ldots, X_K)$ by,

$$Y_n = \sum_{k=1}^{K} \alpha_{n,k} X_k, \quad \sum_{k=1}^{K} \alpha_{n,k}^2 = 1.$$

Y_n denotes the firm's **composite factor** and ϵ_n the **idiosyncratic shock**.
β_n borrower n's sensitivity to systematic risk Y_n.
$\alpha_{n,k}$ describe the dependence of obligor n on a sector k.
Now we can rewrite equation (4) as,

$$D_n = \chi_{\{r_n < t_n\}} \sim B(1, \mathbb{P}(r_n < t_n)),$$

We have $P_n = \mathbb{P}(r_n < t_n)$, $t_n = \Phi^{-1}(P_n)$ and,
$P_n(y_n) \equiv \mathbb{P}(r_n < t_n | Y_n = y_n) = \Phi\left(\frac{t_n - \beta_n y_n}{\sqrt{1-\beta_n^2}}\right)$, cond. default probability.

Conditional Default Probability

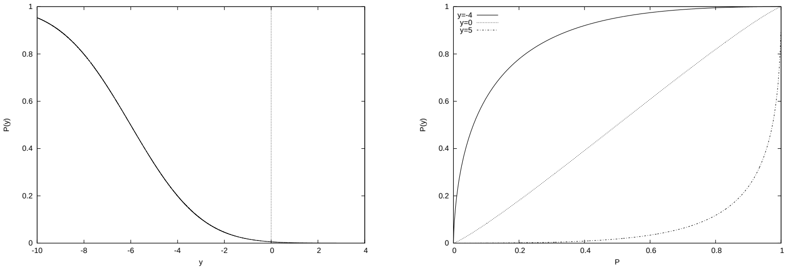

Figure: Conditional default probabilities.

Portfolio Loss

Purpose: find an expression for the portfolio loss variable $\mathcal{L}$.

Assuming a constant loss given default equal to L_n for obligor n, the portfolio loss distribution can then be derived as,

$$\mathbb{P}(\mathcal{L} \leq l) = \sum_{\substack{(d_1,\ldots,d_N)\in\{0,1\}^N \\ \sum_{n=1}^{N} s_n \cdot L_n \cdot d_n \leq l}} \left(\sum_{n=1}^{N} s_n \cdot L_n \cdot d_n \right) \cdot \mathbb{P}(D_1 = d_1, \ldots, D_N = d_N).$$

Impractical from a computational point of view for realistic portfolios (for instance $N = 1000$).

Remark 1.1

We present an analytical approximation for the α^{th} percentile of the loss distribution in the one-factor framework, under the assumption that portfolios are infinitely fine-grained such that the idiosyncratic risk is completely diversified.

Portfolio Loss

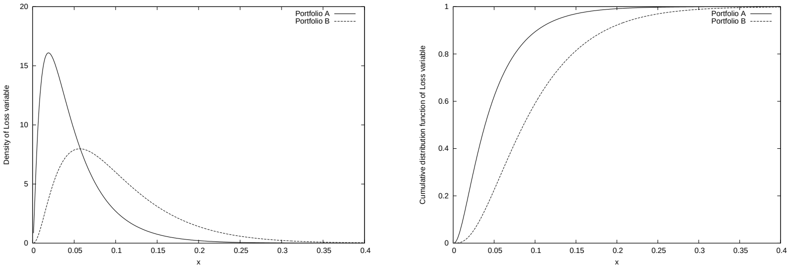

Figure: Densities and distributions for portfolios A and B.

The ASRF Model

The Asymptotic Single Risk Factor Model (ASRF) is the model chosen in Basel II to calculate regulatory capital. It is based on the one-factor Merton model and it mainly relies in the following assumptions,

Assumption 1.3

1. *Portfolios are infinitely fine-grained, i.e. no exposure accounts for more than an arbitrarily small share of total portfolio exposure.*
2. *Dependence across exposures is driven by a single systematic risk factor Y. Default indicators are mutually independent conditional on Y.*

The ASRF Model

Theorem 1.1

Let us denote the exposure share of obligor n by $s_n = \frac{E_n}{\sum_{n=1}^{N} E_n}$*. Then, under assumption 1.3 the portfolio loss ratio* $\mathcal{L} = \sum_{n=1}^{N} s_n \cdot L_n \cdot D_n$ *conditional on any realization y of the systematic risk factor Y satisfies,*

$$\mathcal{L} - \mathbb{E}(\mathcal{L}|Y = y) \to 0 \text{ almost surely as } N \to \infty.$$

Under one-factor Merton model and assuming L_n to be deterministic,

$$\mathbb{E}\left(\mathcal{L}|Y = y\right) = \sum_{n=1}^{N} s_n \cdot L_n \cdot \Phi\left(\frac{t_n - \sqrt{\rho_n} y}{\sqrt{1 - \rho_n}}\right). \quad (5)$$

The ASRF Model

By Theorem 1.1,

$$\mathrm{VaR}_\alpha(\mathcal{L}) - \mathbb{E}(\mathcal{L}|Y = l_{1-\alpha}(Y)) \to 0 \text{ a.s. as } N \to \infty.$$

Finally,

$$\mathrm{VaR}_\alpha^A = \sum_{n=1}^{N} s_n \cdot L_n \cdot \Phi\left(\frac{t_n + \sqrt{\rho_n}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_n}}\right),$$

and,

$$\mathrm{VaRC}_{\alpha,n}^A = s_n \cdot \frac{\partial \mathrm{VaR}_\alpha^A}{\partial s_n} = s_n \cdot L_n \cdot \Phi\left(\frac{t_n + \sqrt{\rho_n}\Phi^{-1}(\alpha)}{\sqrt{1-\rho_n}}\right).$$

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Concentration Risk

However:

- Real world portfolios are not perfectly fine-grained.
- The ASRF model might be approximately valid for huge portfolios but less satisfactory for portfolios of smaller institutions (or more specialized).
- The formula can **underestimate the required economic capital**.
- Does not allow the measurement of sector concentration risk.

In practice: Monte Carlo simulations (robust but **computationally intensive**). The **variance** is an issue.

Proposal: A new method based on wavelets to overcome the computational complexity.

Outline

The Haar System

Consider $f \in L^2(\mathbb{R})$.

$V_j = \{g \in L^2 : g \text{ is constant on } I_{j,k},\ k \in \mathbb{Z}\}$ and $I_{j,k} = [\frac{k}{2^j}, \frac{k+1}{2^j})$.

An orthogonal basis for V_j is given by the family:

$$\phi_{j,k}(x) = 2^{\frac{j}{2}}\phi(2^j x - k),\ k \in \mathbb{Z}, \phi(x) = \chi_{[0,1)}(x).$$

Consider the orthogonal projector onto V_j, $\mathcal{P}_j : L^2(\mathbb{R}) \to V_j$.

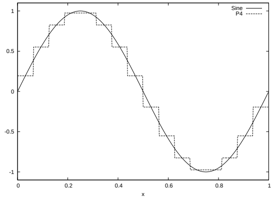
We can write,

$$\mathcal{P}_j f = \sum_{k=-\infty}^{\infty} c_{j,k}\phi_{j,k},$$

$$c_{j,k} = \langle f, \phi_{j,k} \rangle = 2^{\frac{j}{2}} \int_{I_{j,k}} f(x)dx.$$

The Haar System

Example: component $\mathcal{P}_4 f$ on $[0, 1]$, for $f(x) = \sin(2\pi x)$.



The Haar System

ϕ is called the **scaling function** (approximation at certain level)

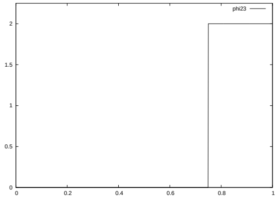


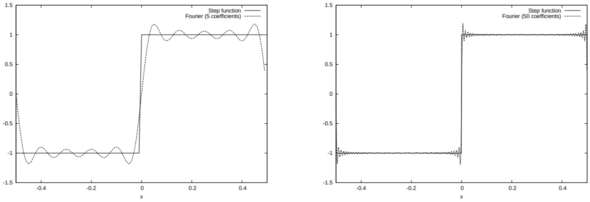
Figure: Scaling ($\phi_{2,3}$) function.

The Haar System

Example: the step function

$$f(x) = \begin{cases} -1, & x \in [-\frac{1}{2}, 0], \\ 1, & x \in (0, \frac{1}{2}], \\ 0, & \text{otherwise.} \end{cases}$$

This function is poorly approximated by its Fourier series:



Wavelets are more flexible: $f(x) = \frac{\sqrt{2}}{2}\phi_{1,0}(x) - \frac{\sqrt{2}}{2}\phi_{1,-1}(x)$.

Laplace Transform

Define the **Laplace transform** of f,

$$\widetilde{f}(s) = \int_0^{+\infty} e^{-sx} f(x)dx = \lim_{\tau \to +\infty} \int_0^{\tau} e^{-sx} f(x)dx, \quad s \in \mathbb{C}.$$

Theorem 2.1

(Bromwich inversion integral) If the Laplace transform of $f(x)$ exists, then,

$$f(x) = \lim_{k \to +\infty} \frac{1}{2\pi i} \int_{\sigma - ik}^{\sigma + ik} \widetilde{f}(s) e^{sx} ds, \quad x > 0,$$

where $|f(x)| \leq e^{\Sigma x}$ for some positive real number Σ and σ is any other real number such that $\sigma > \Sigma$.

The WA Method

Let f be a function in $L^2([0,1])$.

$$f(x) = \lim_{m \to +\infty} f_m(x), \quad f_m(x) = \sum_{k=0}^{2^m-1} c_{m,k}\phi_{m,k}(x),$$

where,

$$c_{m,k} = \int_{\frac{k}{2^m}}^{\frac{k+1}{2^m}} f(x)\phi_{m,k}(x)dx,$$

$k = 0, \ldots, 2^m - 1.$

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The WA Method

Consider the Laplace Transform of f:

$\widetilde{f}(s) = \int_0^{+\infty} e^{-sx} f(x)dx$, (assume $f(x) = 0, \forall x \notin [0,1]$).

Wavelet Approximation (WA) method: **approximate** $\widetilde{f}$ by $\widetilde{f}_m$ and **compute** the coefficients $c_{m,k}$.

$$\widetilde{f}(s) = \int_0^{+\infty} e^{-sx} f(x)dx \simeq \int_0^{+\infty} e^{-sx} f_m(x)dx =$$

$$= \frac{2^{m/2}}{s}\left(1 - e^{-s\frac{1}{2^m}}\right)\sum_{k=0}^{2^m-1} c_{m,k} e^{-s\frac{k}{2^m}}.$$

Change of variable $z = e^{-s\frac{1}{2^m}}$: $\sum_{k=0}^{2^m-1} c_{m,k} z^k \simeq \overline{Q}_m(z)$.

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The WA Method

We obtain the coefficients $c_{m,k}$ by means of the Cauchy's integral formula,

$$c_{m,k} \simeq \frac{2}{\pi r^k} \int_0^{\pi} \Re(Q_m(re^{iu})) \cos(ku) du, \quad k = 0, ..., 2^m - 1.$$

The integral can be evaluated by means of the **trapezoidal rule**,

$$I(r,k) = \int_0^{\pi} \Re(Q_m(re^{iu})) \cos(ku) du,$$

$$I(r,k;h) = \frac{h}{2} \left(Q_m(r) + (-1)^k Q_m(-r) + 2 \sum_{j=1}^{m_T - 1} \Re(Q_m(re^{ih_j})) \cos(kh_j) \right),$$

where $h = \frac{\pi}{m_T}$ and $h_j = jh$ for all $j = 0, \ldots, m_T$.

Then,

$$c_{m,k} \simeq \frac{2}{\pi r^k} I(r,k) \simeq \frac{2}{\pi r^k} I(r,k;h), \quad k = 1, ..., 2^m - 1.$$

Outline

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The Model

Focus on the **one-factor Merton** model. **Assume:**

$$L_n = 100\%, \sum_{n=1}^{N} E_n = 1.$$

Let F be the CDF of $\mathcal{L}$ and $f_{\mathcal{L}}$ its PDF.

$$r_n = \sqrt{\rho}Y + \sqrt{1-\rho}\epsilon_n,$$

(Y, ϵ_n i.i.d. $N(0,1)$).

Conditional default probabilities, $P_n(y) \equiv \Phi\left(\frac{t_n - \sqrt{\rho}y}{\sqrt{1-\rho}}\right)$, $t_n = \Phi^{-1}(P_n)$.

The Approximation

Consider,

$$F(x) = \begin{cases} \overline{F}(x), & \text{if } 0 \leqslant x \leqslant 1, \\ 1, & \text{if } x > 1, \end{cases}$$

Define unconditional MGF: $\tilde{M}_{\mathcal{L}}(s) \equiv \mathbb{E}(e^{-s\mathcal{L}})$.

Assumption 3.1

Conditional Independence Framework. *If the systematic factor Y is fixed, defaults occur independently because the only remaining uncertainty is the idiosyncratic risk.*

Define conditional MGF:
$\bar{\bar{M}}_{\mathcal{L}}(s; y) \equiv \mathbb{E}(e^{-s\mathcal{L}} \mid Y = y) = \prod_{n=1}^{N} \left[1 - P_n(y) + P_n(y) e^{-sE_n}\right]$.

Then:
$\tilde{M}_{\mathcal{L}}(s) = \mathbb{E}(\bar{\bar{M}}_{\mathcal{L}}(s; y)) = \int_{\mathbb{R}} \prod_{n=1}^{N} \left[1 - P_n(y) + P_n(y) e^{-sE_n}\right] \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$

The Approximation

Since: $\overline{F} \in L^2([0,1])$ then:

$\overline{F}(x) \simeq \overline{F}_m(x), \quad \overline{F}_m(x) = \sum_{k=0}^{2^m-1} c_{m,k}\phi_{m,k}(x),$
$\overline{F}(x) = \lim_{m\to+\infty} \overline{F}_m(x).$

Observe: $\tilde{M}_{\mathcal{L}}(s) = \int_0^{+\infty} e^{-sx}F'(x)dx = e^{-s} + s\int_0^1 e^{-sx}\overline{F}(x)dx.$

Then: $\left(\tilde{M}_{\mathcal{L}}(s) - e^{-s}\right)/s$ is the Laplace transform of $\overline{F}$.

Apply the WA method,

Compute: $c_{m,k}$.

VaR Computation

We have: $\overline{F}(\text{VaR}_\alpha) \simeq \overline{F}_m(\text{VaR}_\alpha) = 2^{\frac{m}{2}} \cdot c_{m,\overline{k}}$, $\overline{k} \in \{0, 1, ..., 2^m - 1\}$.

Algorithm:

1. Compute $\overline{F}_m(\frac{2^{m-1}}{2^m}) = 2^{\frac{m}{2}} \cdot c_{m,2^{m-1}}$.
2. If $\overline{F}_m(\frac{2^{m-1}}{2^m}) > \alpha$ then compute $\overline{F}_m(\frac{2^{m-1}-2^{m-2}}{2^m})$, otherwise compute $\overline{F}_m(\frac{2^{m-1}+2^{m-2}}{2^m})$.
3. Finish after m steps storing the $\overline{k}$ value s.t. $\overline{F}_m(\frac{\overline{k}}{2^m})$ is the closest value to α.

In fact, $\overline{F}_m(\xi) = \overline{F}_m(\frac{\overline{k}}{2^m})$, for all $\xi \in \left[\frac{\overline{k}}{2^m}, \frac{\overline{k}+1}{2^m}\right)$.

So take: $\text{VaR}_\alpha^{W(m)} = \frac{2\overline{k}+1}{2^{m+1}}$.

Parameters: $m = 10, m_T = 2^m, l = 20.$ MC with 5×10^6 scenarios.

Portfolio 3.1

We consider $N = 102$ *obligors, with* $P_n = 0.1\%$, $E_n = 1, n = 1, \ldots, 100$, $E_{101} = E_{102} = 20$, $\rho = 0.3$ *and* $L_n = 1$.

Method	$\text{VaR}_{0.999}$	Relative Error
Monte Carlo	0.1500	
ASRF	0.0474	−68.39%
Saddle Point	0.1270	−15.37%
Wavelet Approximation	0.1490	−0.69%

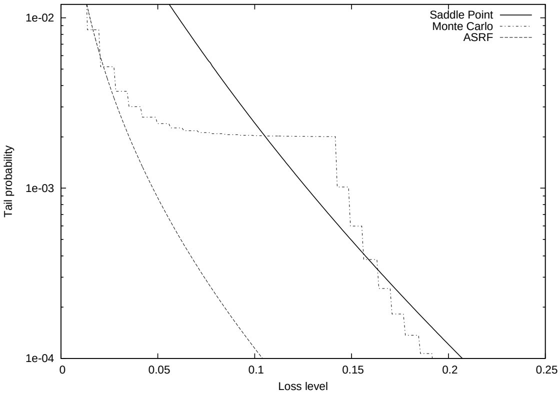

Figure: Tail probability approximation of a heterogeneous portfolio with severe name concentration.

Portfolio	N	P_n	E_n	ρ	HHI	$\frac{1}{N}$
P1	100	0.21%	$\frac{C}{n}$	0.15	0.0608	0.0100
P2	1000	1.00%	$\frac{C}{n}$	0.15	0.0293	0.0010
P3	1000	0.30%	$\frac{C}{n}$	0.15	0.0293	0.0010
P4	10000	1.00%	$\frac{C}{n}$	0.15	0.0172	0.0001
P5	20	1.00%	$\frac{1}{N}$	0.5	0.0500	0.0500
P6	10	0.21%	$\frac{C}{n}$	0.5	0.1806	0.1000

Portfolio	$\text{VaR}_{0.999}^{W(8)}$	$\overline{\text{RE}}(0.999, 8)$	$\text{VaR}_{0.999}^{W(9)}$	$\overline{\text{RE}}(0.999, 9)$	$\text{VaR}_{0.999}^{W(10)}$	$\overline{\text{RE}}(0.999, 10)$	$\text{VaR}_{0.999}^{M}$
P1	0.1934	−0.19%	0.1963	1.32%	0.1938	0.06%	0.1937
P2	0.1934	1.01%	0.1924	0.50%	0.1919	0.25%	0.1914
P3	0.1426	1.46%	0.1416	0.77%	0.1411	0.42%	0.1405
P4	0.1621	0.24%	0.1611	−0.36%	0.1616	−0.06%	0.1617

Portfolio	$\text{VaR}_{0.999}^{W(8)}$	$\text{VaR}_{0.999}^{W(9)}$	$\text{VaR}_{0.999}^{W(10)}$	$\text{VaR}_{0.999}^{M}$
P1	0.2	0.4	0.7	58.3
P2	1.8	3.6	7.2	571.6
P3	1.8	3.6	7.2	567.6
P4	18.2	36.1	72.4	1379.1

Table: CPU time (in seconds).

		2^{10}		
Portfolio	$\text{VaR}_{0.9999}^{W(10)}$	$\overline{\text{RE}}(0.9999, 10)$	$\text{VaR}_{0.99999}^{W(10)}$	$\overline{\text{RE}}(0.99999, 10)$
P1	0.2251	−0.07%	0.2935	−1.70%
P2	0.2622	−0.46%	0.3325	−1.80%
P3	0.1812	−0.10%	0.2290	−1.88%
P4	0.2261	−0.25%	0.2935	−1.30%
		2^{11}		
Portfolio	$\text{VaR}_{0.9999}^{W(10)}$	$\overline{\text{RE}}(0.9999, 10)$	$\text{VaR}_{0.99999}^{W(10)}$	$\overline{\text{RE}}(0.99999, 10)$
P1	0.2251	−0.07%	0.2935	−1.70%
P2	0.2622	−0.46%	0.3325	−1.80%
P3	0.1812	−0.10%	0.2290	−1.88%
P4	0.2261	−0.25%	0.2935	−1.30%
		MC		
Portfolio	$\text{VaR}_{0.9999}^{M}$		$\text{VaR}_{0.99999}^{M}$	
P1	0.2253		0.2985	
P2	0.2634		0.3386	
P3	0.1813		0.2334	
P4	0.2267		0.2973	

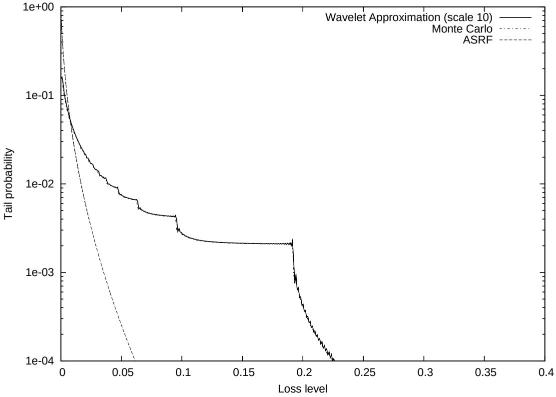

Figure: Tail probability approximation of Portfolios P1 at scale $m = 10$.

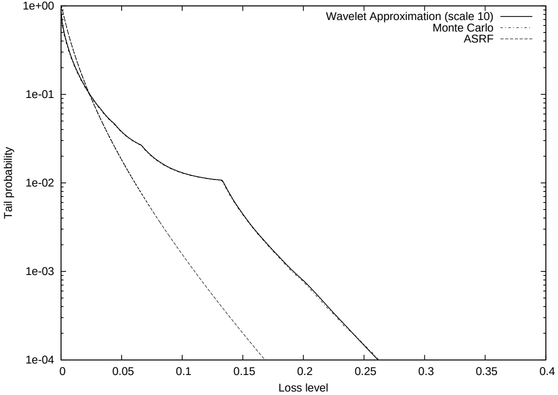

Figure: Tail probability approximation of Portfolios P2 at scale $m = 10$.

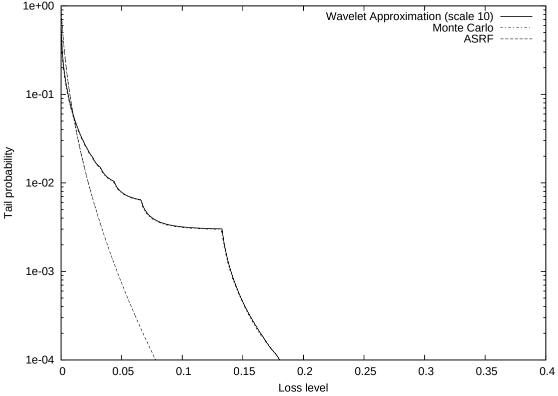

Figure: Tail probability approximation of Portfolios P3 at scale $m = 10$.

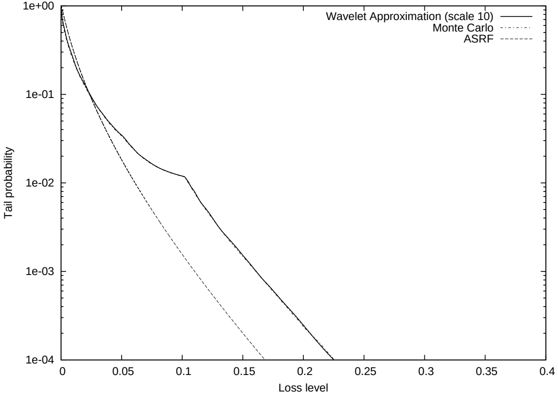

Figure: Tail probability approximation of Portfolios P4 at scale $m = 10$.

Outline

The Expected Shortfall

By definition: $\mathrm{ES}_\alpha(E) = \frac{1}{1-\alpha}\int_{VaR_\alpha(E)}^{+\infty} x f_{\mathcal{L}}(E,x)dx$.

Integrating by parts,

$$\mathrm{ES}_\alpha(E) = \frac{1}{1-\alpha}\left(1 - \alpha \mathrm{VaR}_\alpha(E) - \int_{VaR_\alpha(E)}^{1} \overline{F}(E,x)dx\right) \simeq \mathrm{ES}_\alpha^{W(m)}(E),$$

where,

$$\mathrm{ES}_\alpha^{W(m)}(E) \equiv \frac{1}{1-\alpha}\left(1 - \alpha \mathrm{VaR}_\alpha^{W(m)}(E) - \frac{1}{2^{\frac{m}{2}+1}} c_{m,\overline{k}}(E) - \frac{1}{2^{\frac{m}{2}}} \sum_{k=\overline{k}+1}^{2^m-1} c_{m,k}(E)\right)$$

VaR Contributions

Taking into account that $\overline{F}(E, \text{VaR}_\alpha) = \alpha$ then,

$$\text{VaRC}_{\alpha,i} \equiv E_i \cdot \frac{\partial \text{VaR}_\alpha}{\partial E_i} = -E_i \cdot \frac{\frac{\partial \overline{F}(E,\text{VaR}_\alpha)}{\partial E_i}}{f_{\mathcal{L}}(E, \text{VaR}_\alpha)}. \tag{6}$$

Recall that: $\overline{F}(E, x) \simeq \sum_{k=0}^{2^m-1} c_{m,k}(E)\phi_{m,k}(x)$.

Numerator: $\frac{\partial \overline{F}(E,x)}{\partial E_i} \simeq \sum_{k=0}^{2^m-1} \frac{\partial c_{m,k}(E)}{\partial E_i}\phi_{m,k}(x)$.

Evaluating this expression in VaR_α: $\frac{\partial \overline{F}(E,\text{VaR}_\alpha)}{\partial E_i} \simeq 2^{\frac{m}{2}} \frac{\partial c_{m,\overline{k}}(E)}{\partial E_i}$.

Finally, $\text{VaRC}_{\alpha,i} \simeq \text{VaRC}_{\alpha,i}^{W(m)}$, where $\text{VaRC}_{\alpha,i}^{W(m)} \equiv \mathcal{C} \cdot E_i \cdot \frac{\partial c_{m,\overline{k}}(E)}{\partial E_i}$ and $\mathcal{C}$ is a constant such that $\sum_{i=1}^{N} \text{VaRC}_{\alpha,i}^{W(m)} = \text{VaR}_\alpha^{W(m)}$.

ES Contributions

Take partial derivatives w.r.t. the exposures,

$$\begin{aligned}\mathrm{ESC}_{\alpha,i} &\equiv E_i \cdot \frac{\partial \mathrm{ES}_\alpha}{\partial E_i} \\ &= E_i \cdot \frac{1}{1-\alpha}\left(-\alpha \frac{\partial \mathrm{VaR}_\alpha}{\partial E_i} + \frac{\partial \mathrm{VaR}_\alpha}{\partial E_i}\overline{F}(E, \mathrm{VaR}_\alpha) - \int_{\mathrm{VaR}_\alpha}^{1} \frac{\partial \overline{F}(E,x)}{\partial E_i} dx\right) \\ &\simeq \mathrm{ESC}_{\alpha,i}^{W(m)},\end{aligned}$$

where,

$$\mathrm{ESC}_{\alpha,i}^{W(m)} \equiv -E_i \cdot \frac{1}{2^{\frac{m}{2}}} \cdot \frac{1}{1-\alpha} \cdot \left(\frac{1}{2}\frac{\partial c_{m,\overline{k}}(E)}{\partial E_i} + \sum_{k=\overline{k}+1}^{2^m-1} \frac{\partial c_{m,k}(E)}{\partial E_i}\right).$$

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Parameters: $m = 10, m_T = 2^m$. Contributions: MC with 10^8 scenarios.

Portfolio 4.1

This portfolio has $N = 10000$ obligors with $\rho = 0.15$, $P_n = 0.01$ and $E_n = \frac{1}{n}$ for $n = 1, ..., N$, as Portfolio P4.

Method	$l/2 - n_2$	$l/2 - n_1$	$\alpha = 0.9999$	$\alpha = 0.99999$
VaR_α^M			0.2267	0.2973
VaR_α^A			0.1683 (−25.76%)	0.2322 (−21.91%)
$\text{VaR}_\alpha^{W(10)}(20)$	10	10	0.2261 (−0.25%)	0.2935 (−1.30%)
$\text{VaR}_\alpha^{W(10)}(20, 4 \cdot 10^{-1})$	9	0	0.2261 (−0.25%)	0.2944 (−0.97%)
$\text{VaR}_\alpha^{W(10)}(20, 6 \cdot 10^{-1})$	7	0	0.2261 (−0.25%)	0.2944 (−0.97%)

Method	$l/2 - n_2$	$l/2 - n_1$	$\alpha = 0.99$	$\alpha = 0.999$	$\alpha = 0.9999$
ES_α^M			0.1290	0.1895	0.2553
$\text{ES}_\alpha^{W(10)}(20)$	10	10	0.1290 (−0.02%)	0.1895 (−0.01%)	0.2556 (0.12%)
$\text{ES}_\alpha^{W(10)}(20, 4 \cdot 10^{-1})$	9	0	0.1289 (−0.12%)	0.1895 (−0.01%)	0.2556 (0.12%)
$\text{ES}_\alpha^{W(10)}(20, 6 \cdot 10^{-1})$	7	0	0.1289 (−0.11%)	0.1896 (0.00%)	0.2559 (0.25%)

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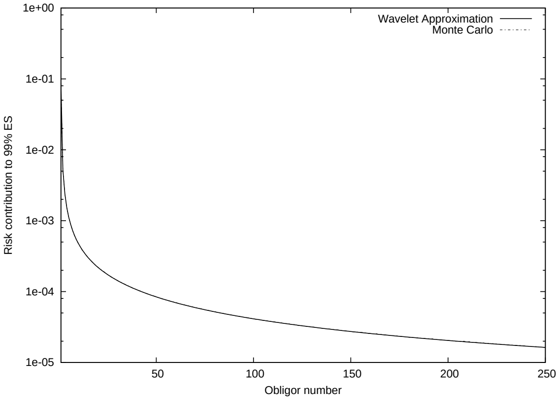

Figure: The 250 biggest risk contributions to the ES at 99% confidence level for the Portfolio 4.1.

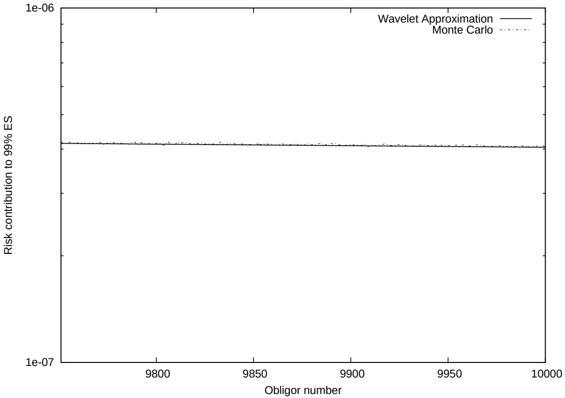

Figure: The 250 smallest risk contributions to the ES at 99% confidence level for the Portfolio 4.1.

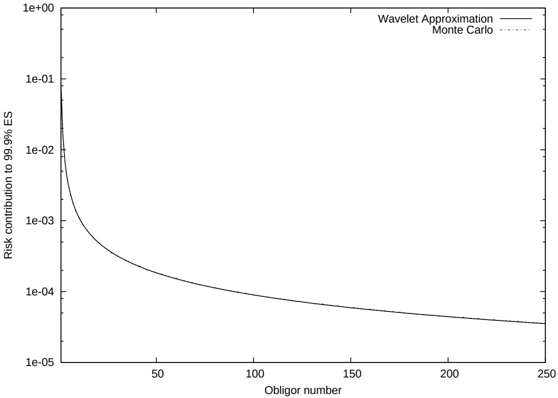

Figure: The 250 biggest risk contributions to the ES at 99.9% confidence level for the Portfolio 4.1.

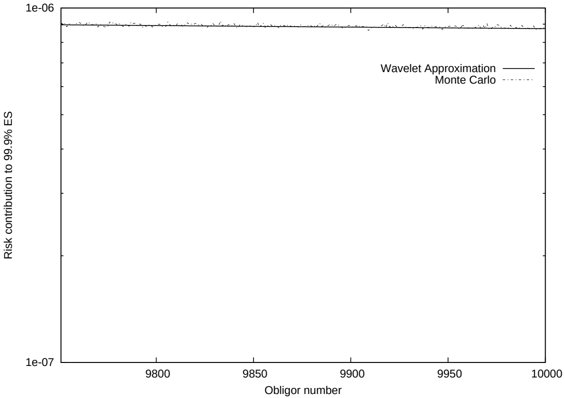

Figure: The 250 smallest risk contributions to the ES at 99.9% confidence level for the Portfolio 4.1.

Method	$l/2-\overline{n}_2$	$l/2-\overline{n}_1$	$\alpha=0.99$	$\alpha=0.999$
$\sum_{n=1}^{N}\text{ESC}_{\alpha,n}^{M}$			0.1290	0.1892
$\sum_{n=1}^{N}\text{ESC}_{\alpha,n}^{W(10)}(20)$	10	10	0.1293	0.1891
$\sum_{n=1}^{N}\text{ESC}_{\alpha,n}^{W(10)}(20,10^{-4})$	10	4	0.1293	0.1890

Method	CPU time (in seconds)
$\text{VaR}_{\alpha}^{W(10)}(20,6\cdot10^{-1})$	25.3
$\text{VaR}_{\alpha}^{W(9)}(20,6\cdot10^{-1})$	12.7
$\text{VaR}_{\alpha}^{W(8)}(20,6\cdot10^{-1})$	6.4
$\text{ESC}_{\alpha,n}^{W(10)}(20,10^{-4})$	622.5

Portfolio 4.2

This portfolio has $N = 1001$ obligors, with $E_n = 1$ for $n = 1, ..., 1000$, and one obligor with $E_{1001} = 100$. $P_n = 0.0033$ for all the obligors and $\rho = 0.2$.

Method	CPU time (in seconds)
$\text{VaR}_{\alpha}^{W(10)}(64, 5 \cdot 10^{-1})$	5.6
$\text{ESC}_{\alpha,n}^{W(10)}(64, 10^{-4})$	78.4

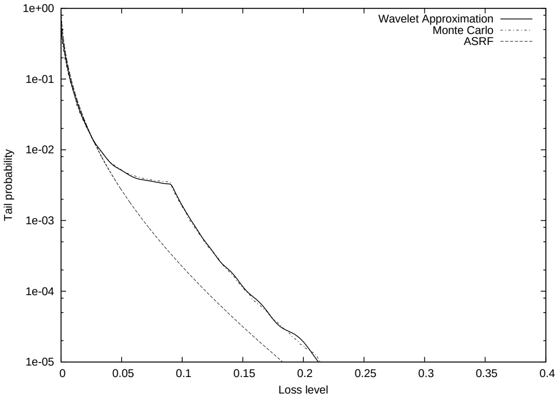

Figure: Tail probability approximation for Portfolio 4.2.

Portfolio 4.3

This portfolio has $N = 100$ obligors, all them with $P_n = 0.01$, $\rho = 0.5$ and exposures (as in Glasserman (2005)),

$$E_n = \begin{cases} 1, & n = 1, ..., 20, \\ 4, & n = 21, ..., 40, \\ 9, & n = 41, ..., 60, \\ 16, & n = 61, ..., 80, \\ 25, & n = 81, ..., 100, \end{cases}$$

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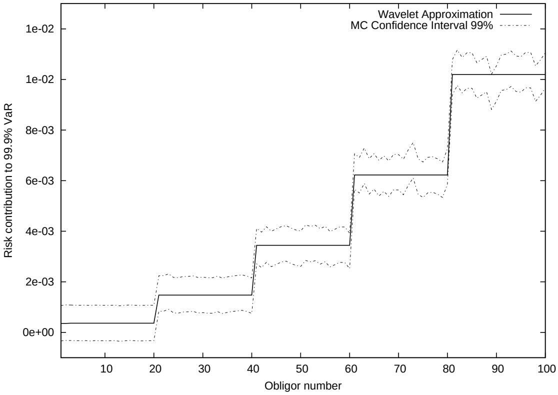

Figure: VaR contributions at 99.9% confidence level for Portfolio 4.3 using the WA method and GH integration formulas with 64 nodes and $\overline{\epsilon} = 10^{-4}$.

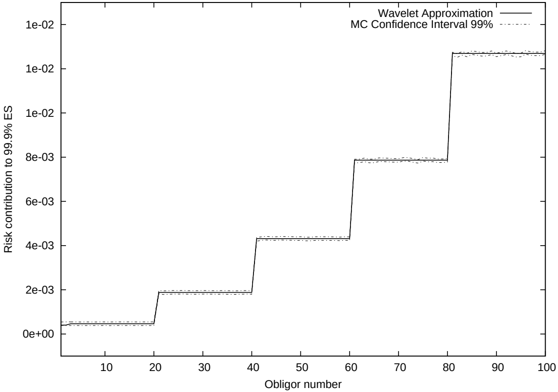

Figure: Expected Shortfall contributions at 99.9% confidence level for Portfolio 4.3 using the WA method and GH integration formulas with 64 nodes and $\overline{\epsilon} = 10^{-4}$.

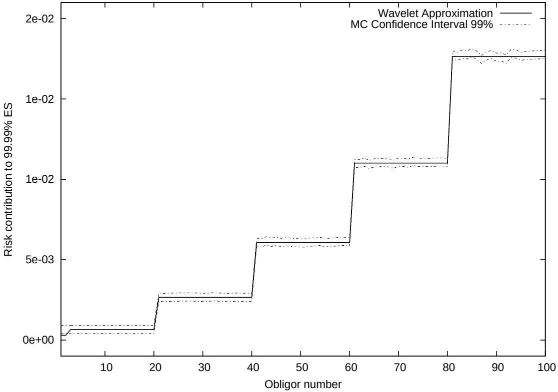

Figure: Expected Shortfall contributions at 99.99% confidence level for Portfolio 4.3 using the WA method and GH integration formulas with 64 nodes and $\overline{\epsilon} = 10^{-4}$.

Outline

1. Portfolio Credit Risk Modeling
2. Haar Wavelets for Laplace Transform Inversion
3. The WA Method to Quantify Losses
4. Credit Risk Contributions
5. The WA Extension to the Multi-Factor Model
6. Conclusions

The Model

Multi-factor Gaussian copula:

$$W_n = \boldsymbol{a_n}^T \boldsymbol{Y} + b_n Z_n. \tag{7}$$

Multi-factor t-copula:

$$W_n = \sqrt{\frac{\nu}{V}} \left(\boldsymbol{a_n}^T \boldsymbol{Y} + b_n Z_n \right). \tag{8}$$

The computation of characteristic functions (or Laplace transforms) are not affordable by numerical quadrature.

Outline

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Conclusions

- New method for Laplace Transform inversion based on Haar wavelets. Particularly well suited for stepped-shape functions, where other state-of-the-art methods fail.
- Computation of the VaR and the ES risk measures under the one-factor Merton model. Very accurate and fast, even in the presence of severe name concentration. **Rel. err.** $< 1\%$.
- The WA method computes the entire distribution of losses without extra computational time (CDO pricing).
- Accurately computation of the risk contributions to the VaR and the ES. **Rel. err.** $< 1\%$.
- Extension to the multi-factor Merton model (to account for sector concentration).

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- J.J. Masdemont and L. Ortiz-Gracia (2014). Haar wavelets-based approach for quantifying credit portfolio losses. **Quantitative Finance**, 14(9), 1587–1595.
- L. Ortiz-Gracia and J.J. Masdemont (2014). Credit risk contributions under the Vasicek one-factor model: a fast wavelet expansion approximation. **Journal of Computational Finance**, 17(4), 59–97.
- G. Colldeforns-Papiol, L. Ortiz-Gracia and C.W. Oosterlee (2017). Quantifying credit portfolio losses under multi-factor models. **Submitted for publication**, available at `www.ssrn.com`.